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2000 J. Phys. A: Math. Gen. 33 L235

(http://iopscience.iop.org/0305-4470/33/27/101)

J. Phys. A: Math. Gen. **33** (2000) L235–L241. Printed in the UK PII: S0305-4470(00)12232-2

LETTER TO THE EDITOR

Tsallis statistics and fully developed turbulence

T Arimitsu† and N Arimitsu‡

† Institute of Physics, University of Tsukuba, Ibaraki 305-8571, Japan
‡ Department of Computer Engineering, Yokohama National University, Kanagawa 240-8501, Japan

Received 1 March 2000

Abstract. An analysis of fully developed turbulence is developed based on the assumption that the underlying statistics of the system is that of the Tsallis ensemble. The multifractal spectrum $f_{\mathrm{T}}(\alpha)$ corresponding to the Tsallis-type distribution function is determined self-consistently in the sense that all parameters can be obtained through the observed value of the intermittency exponent. It is shown that the scaling exponents ζ_m of the velocity structure function derived with the help of the multifractal spectrum fit very well with experimental data. It is revealed that the asymptotic expression of ζ_m for $m \gg 1$ has a log term. The present self-consistent approach narrowed down the value of intermittency exponent μ for the fully developed turbulence to $\mu = 0.235 \pm 0.015$.

In a previous paper [1], we showed that the Tsallis index q [2, 3]† corresponding to the p-model [5] can be effectively determined by observed values of the intermittency exponent μ with the help of the scaling relation (14) below [6]. We proposed in [1] a Tsallis-type distribution function for the probability density function of the local dissipation, and revealed that the proposed distribution function with the Tsallis index q determined by the observed value μ fits very well with the binomial distribution function of the p-model.

In this letter, we develop the program in [1] much further with the assumption that the underlying statistics of the system of fully developed turbulence is that of the Tsallis ensemble. The ensemble is a generalized one with a non-extensive character (see (11) below) including the Boltzmann–Gibbs (extensive) ensemble as a special case. This may give us an interpretation of the question of why the multifractal analysis (p-model) works, based upon the statistical mechanical background in the sense that an appropriate probability density function of the local dissipation is derived by taking the extremal of the Tsallis entropy [2–4] under two constraints, i.e. the normalization of probability and the quantity related to the intermittency exponent being constant. Note that most of the theories produced up to now have been constructed on extensive statistics, and others do not have a statistical mechanical basis. We expect that the present approach will also provide us with a plausible understanding of the somewhat unfamiliar Tsallis statistics itself as well as of an old but still new difficult problem related to the intermittency in fully developed turbulence.

We will determine the multifractal spectrum $f_{\mathrm{T}}(\alpha)$ corresponding to the Tsallis-type distribution function self-consistently in the sense that all parameters can be calculated by using the observed value of the intermittency exponent. There is no other fitting parameter. With the multifractal spectrum, we will derive the scaling exponents ζ_m of velocity structure function, and compare them with experimental data and with the curves given by other theories,

† For an updated bibliography on the subject see [4].

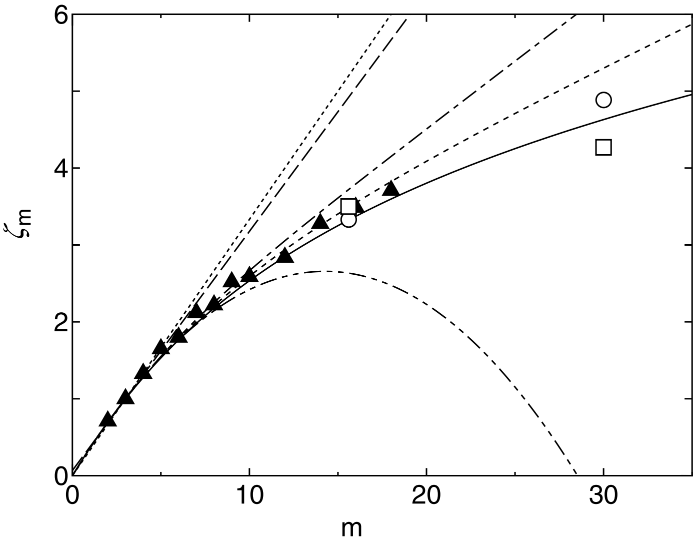

Figure 1. Scaling exponents ζ_m of velocity structure functions. The present result for $\mu = 0.235$ is given by the solid curve. The solid triangles are the experimental results by Anselmet *et al* [11]; the squares and the circles are from [5]. K41 is given by the dotted line, the β-model ($D_\beta = 2.8$) by the dashed line, the p-model ($\mu = 0.235$) by the dotted–dashed curve, the log-Poisson model by the short-dashed curve and the log-normal model ($\mu = 0.235$) by the double-dotted–dashed curve.

i.e. K41, log normal, β-model, p-model and log Poisson. We will show that the present result fits very well with all the experimental data (see figure 1). We also find that there is a log term in ζ_m for $m \gg 1$.

The study of fully developed turbulence was started by Kolmogorov [7] by dimensional analysis with the assumption that any physical mean values are determined by the kinetic viscosity, $\nu = \eta/\rho$, and the energy input (output) rate, ϵ. Here, ρ and η represent, respectively, mass density and static viscosity. In the energy input range, since ν may not take part, the size ℓ_0 of the grid which produces turbulence should be given by $\ell_0 = u_0^3/\epsilon$, with the velocity u_0 of fluid at the grid. On the other hand, in the dissipation range, as ν plays the leading part, the typical size ℓ_d of the range is determined by $\ell_d = (\nu^3/\epsilon)^{1/4}$. Note that the Reynolds number Re of the system is, then, given by $Re = u_0\ell_0/\nu = (\ell_0/\ell_d)^{4/3}$.

For the high-Re limit $Re \gg 1$, there exists a wide inertial range, which is characterized by the size $\ell_n = \ell_0\delta_n$, $\delta_n = 1/2^n$ $(n = 0, 1, 2, \ldots)$ of eddies satisfying $\ell_0 \gg \ell_n \gg \ell_d$, and the Navier–Stokes equation, $\partial\vec{u}/\partial t + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}(p/\rho) + \nu\nabla^2\vec{u}$, is invariant under the scale transformation [8]: $\vec{r}' = \lambda\vec{r}$, $\vec{u}' = \lambda^{\alpha/3}\vec{u}$, $t' = \lambda^{1-\alpha/3}t$, $(p/\rho)' = \lambda^{2\alpha/3}(p/\rho)$. The rate of transfer of energy ϵ_r per unit mass averaged over a domain $r \sim \ell_n$, called the local dissipation of turbulent kinetic energy, behaves as $\epsilon_r \sim \delta u_r^3/r \propto r^{\alpha-1}$. The total dissipation E_r occurring in a box of size r will be

$$E_r \sim \epsilon_r r^d \propto r^{\alpha-1+d} \tag{1}$$

where d represents the dimension of physical space.

We will restrict ourselves in this paper to the analysis of the measured time series of the streamwise velocity component of an isotropic turbulence behind grids. Then, the dimension of physical space d will be unity [8]. Within the Taylor frozen flow hypothesis, our main interest is the scaling exponents ζ_m of the mth-order velocity correlation of the difference, i.e.

$$\langle(\delta u(r))^m\rangle \propto r^{\zeta_m} \tag{2}$$

with $\delta u(r) = |u(x+r) - u(x)|$ where u is a component of the velocity field $\vec{u}$. The expectation $\langle\cdots\rangle$ is taken by an appropriate probability distribution function, which we will analyse in this paper.

In the inertial range it is assumed that physical quantities are determined by ϵ and $r \sim \ell_n$. Then, within dimensional analysis we see that $\zeta_m = m/3$ in this range. ζ_2 gives the Kolmogorov spectrum [7].

Now, let us assume that the dissipation of turbulent energy is a multifractal. Dividing the d-dimensional space into boxes of size r, and summing powers of different order α of E_r over all boxes, we expect these sums to scale with the size of boxes r according to [8]

$$\sum_\alpha E_r^{\bar{q}} \sim r^{(\bar{q}-1)D_{\bar{q}}} \tag{3}$$

where $D_{\bar{q}}$ is called the generalized dimension (the Renyi dimension). Substituting (1) into (3), and replacing the sum by an integration†: $\sum_\alpha \cdots = \int \mathrm{d}\alpha\, \rho(\alpha) r^{-f_d(\alpha)} \cdots$, we can extract the formulae [8]

$$f_d(\alpha) = \alpha\bar{q} + \tau_d(\bar{q}) \tag{4}$$

with the mass exponent

$$\tau_d(\bar{q}) = (1-\bar{q})D_{\bar{q}} + (d-1)\bar{q} \tag{5}$$

and

$$\alpha = -\,\mathrm{d}\tau_d(\bar{q})/\mathrm{d}\bar{q} \tag{6}$$

satisfied in the limit of small r by making use of the steepest-descent method. These equations determine $f_d(\alpha)$ and α when $D_{\bar{q}}$ is known. Note that $\bar{q}$ is given by

$$\bar{q} = \mathrm{d}f_d(\alpha)/\mathrm{d}\alpha. \tag{7}$$

Equation (4) with (6) or with (7) constitutes the Legendre transformations. We use in this paper the notation $\bar{q}$ to avoid any confusion with the Tsallis index q.

The probability density function $P_\epsilon(\epsilon_r)$ of the local dissipation of turbulent kinetic energy is given by [8]

$$\begin{aligned} P_\epsilon(\epsilon_r)\,\mathrm{d}(\epsilon_r/\epsilon) &\propto \left(\frac{r}{\ell_0}\right)^{D_0-f_d(\alpha)} \frac{\epsilon}{\epsilon_r \ln(r/\ell_0)}\,\mathrm{d}(\epsilon_r/\epsilon) \\ &= \delta_n^{D_0-f_d(\alpha)}\,\mathrm{d}\alpha \\ &= \exp\{[D_0 - f_d(\alpha)]\ln\delta_n\}\,\mathrm{d}\alpha. \end{aligned} \tag{8}$$

It was shown that the intermittency exponent μ is determined by [8]

$$\mu = 1 - D_2. \tag{9}$$

Tsallis [2–4] introduced the non-extensive entropy

$$S_q = \left(\sum_i p_i^q - 1\right)\Big/(1-q) \tag{10}$$

to produce a generalized Boltzmann–Gibbs statistics. The non-extensivity is shown by the pseudo-additivity property [3]

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \tag{11}$$

† $\rho(\alpha)r^{-f_d(\alpha)}$ is the weight from the number of boxes for which α takes on values between α and $\alpha + \mathrm{d}\alpha$. $f_d(\alpha)$ is the multifractal spectrum of the set with the scaling exponent lying between these values.

A and B being probabilistically independent.

By taking the extremal of (10) with the constraint indicating the conservation of probability, $\sum_i p_i = 1$, and with that fixing the q-averaged internal energy [9], $U_q = \sum_i p_i^q E_i / \sum_j p_j^q$, one obtains the general form of the probability distribution function of the Tsallis ensemble in the form

$$p_i = \left[1 - \frac{(1-q)\beta(E_i - U_q)}{\sum_j p_j^q}\right]^{1/(1-q)} \Bigg/ \bar{Z}_q \tag{12}$$

with the partition function

$$\bar{Z}_q = \sum_i \left[1 - \frac{(1-q)\beta(E_i - U_q)}{\sum_j p_j^q}\right]^{1/(1-q)}. \tag{13}$$

Note that Tsallis statistics reduces to Boltzmann–Gibbs statistics taking the limit $q \to 1$. Here, we are using the units where the Boltzmann constant is unity.

It was shown [6] that the value q of the parameter appearing in Tsallis statistics is related to the extremes $\alpha_{\max}$ and $\alpha_{\min}$ of the multifractal spectrum $f_d(\alpha)$ by

$$1/(1-q) = 1/\alpha_{\min} - 1/\alpha_{\max}. \tag{14}$$

Now, we assume that the probability density function can be given by the Tsallis-type distribution function of the form

$$P_{\mathrm{T}}(\alpha)\,\mathrm{d}\alpha = Z_{\mathrm{T}}^{-1}\left[1 - \left(\frac{1-q}{n}\right)\frac{(\alpha-\alpha_0)^2 \ln \delta_n^{-1}}{2X}\right]^{n/(1-q)} \mathrm{d}\alpha \tag{15}$$

with the obvious partition function Z_{T}. The parameters α_0, X and Tsallis index q should be determined by the intermittency exponent μ. With the help of (8), we see that the multifractal spectrum corresponding to the distribution function is given by

$$f_{\mathrm{T}}(\alpha) = D_0 + \frac{1}{1-q}\log_2\left[1 - \left(\frac{1-q}{n}\right)\frac{(\alpha-\alpha_0)^2 \ln \delta_n^{-1}}{2X}\right]. \tag{16}$$

Note that the reason for the subscript in α_0 is because it is defined by $\mathrm{d}f_{\mathrm{T}}(\alpha)/\mathrm{d}\alpha|_{\alpha=\alpha_0} = 0$, indicating that $\alpha_0 = \alpha(\bar{q} = 0)$ (see (7)).

The relation between $\bar{q}$ and α is given by (7), which is solved to give us

$$\alpha_{\bar{q}} - \alpha_0 = \frac{1-\sqrt{\mathcal{D}}}{\bar{q}(1-q)\ln 2} \tag{17}$$

with

$$\sqrt{\mathcal{D}} = \sqrt{1 + 2\bar{q}^2(1-q)X\ln 2}. \tag{18}$$

Then, we have from (5) for $d = 1$

$$\begin{aligned}\tau(\bar{q}) &= (1-\bar{q})D_{\bar{q}} = f_{\mathrm{T}}(\alpha_{\bar{q}}) - \alpha_{\bar{q}}\bar{q} \\ &= 1 - \alpha_{\bar{q}}\bar{q} + \delta\tau(\bar{q})\end{aligned} \tag{19}$$

with

$$\delta\tau(\bar{q}) = \frac{1}{1-q}\log_2\left[1 - \frac{(1-\sqrt{\mathcal{D}})^2}{2\bar{q}^2(1-q)X\ln 2}\right] \tag{20}$$

where we put $f_{\mathrm{T}}(\alpha_0) = D_0 = 1$ for the fractal dimension of the multifractal set. In the case $q \neq 1$, we have

$$\delta\tau(\bar{q}) \to \frac{-1}{1-q}\left[\log_2|\bar{q}| + \log_2\sqrt{X(1-q)\ln 2/2} + \mathcal{O}(1/\bar{q})\right] \tag{21}$$

Table 1. Parameters q, α_0 and X for several values of μ.

	μ	q	α_0	X
a	0.175	0.246	1.10	0.205
b	0.200	0.270	1.12	0.238
c	0.225	0.272	1.13	0.273
d	0.235	0.370	1.14	0.280
e	0.250	0.446	1.14	0.294
f	0.275	0.447	1.16	0.329
g	0.300	0.447	1.18	0.365

for $|\bar{q}| \to \infty$. It should be noted that a log term appears in $\tau(\bar{q})$ for large $|\bar{q}|$†. $\alpha_{\max} = \alpha(\bar{q} = -\infty)$ and $\alpha_{\min} = \alpha(\bar{q} = +\infty)$ are given by

$$\alpha_{\max} - \alpha_0 = \alpha_0 - \alpha_{\min} = \sqrt{2X/((1-q)\ln 2)}. \tag{22}$$

In order to determine three parameters, we need three independent equations. Putting $\bar{q} = 1$ in (19), i.e.

$$\tau(1) = 0 \tag{23}$$

we have the first equation which relates X, q and α_0. Substituting $\bar{q} = 2$ into (19) and using it for (9), i.e.

$$\mu = 1 + \tau(2) \tag{24}$$

we have the second formula, which gives the intermittency exponent μ in terms of X, q and α_0. Substituting the solutions

$$\alpha_- = \alpha_0 - \sqrt{2bX} \qquad \alpha_+ = \alpha_0 + \sqrt{2bX} \tag{25}$$

of $f_{\mathrm{T}}(\alpha) = 0$ with $b = (1 - 2^{-(1-q)})/[(1-q)\ln 2]$ into (14) by replacing $\alpha_{\min}$ and $\alpha_{\max}$ by α_- and α_+, respectively, i.e.

$$1/(1-q) = 1/\alpha_- - 1/\alpha_+ \tag{26}$$

we obtain the third relation between X, q and α_0, which can be solved as

$$\sqrt{2X} = \left[\sqrt{\alpha_0^2 + (1-q)^2} - (1-q)\right] \Big/ \sqrt{b} \tag{27}$$

or

$$\alpha_0 = \sqrt{2bX + 2(1-q)\sqrt{2bX}}. \tag{28}$$

Once we know the value of the intermittency exponent μ, the above three equations (23), (24) and (26) completely determine the three quantities X, q and α_0.

For $\mu = 0.235$ [8], we have $q = 0.370$, $\alpha_0 = 1.14$, $X = 0.280$ (case d in table 1). Then, we obtain $\alpha_+ - \alpha_0 = \alpha_0 - \alpha_- = 0.673$, $\alpha_{\max} - \alpha_0 = \alpha_0 - \alpha_{\min} = 1.133$ and $\bar{q}(\alpha_-) = -\bar{q}(\alpha_+) = 3.72$.

The scaling exponents ζ_m of velocity structure functions given by [8]

$$\zeta_m = 1 - \tau(m/3) = (m/3 - 1)D_{m/3} + 1 \tag{29}$$

† After finishing this letter, the authors were notified that a logarithmic term appears in the multifractal analysis based on the generalized Cantor set proposed by Hosokawa [10]. Its interpretation in terms of the present statistical mechanical approach, i.e. the distribution function, derived by taking the extremal of the generalized entropy, is one of the attractive future problems.

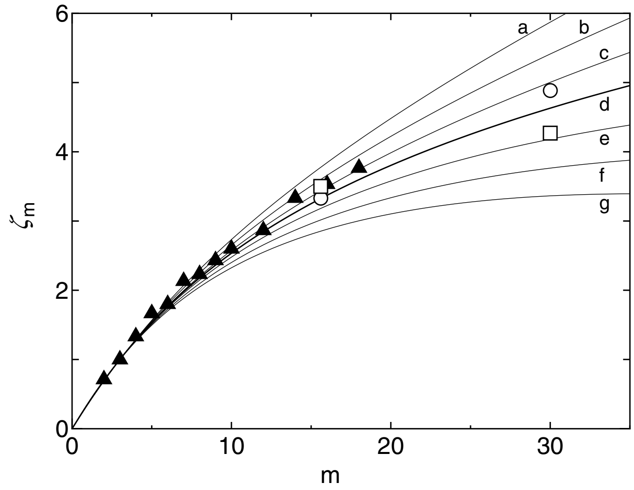

Figure 2. Scaling exponents ζ_m for the cases in table 1.

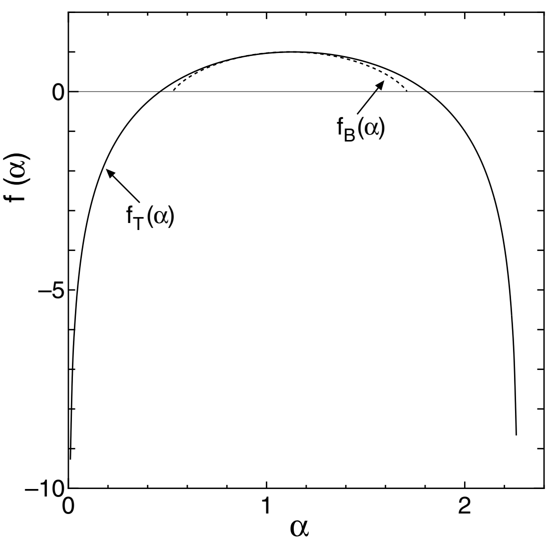

Figure 3. The multifractal spectrum $f_T(\alpha)$ based on Tsallis statistics, and $f_B(\alpha)$ based on the binomial multiplicative process (p-model), for the case $\mu = 0.235$.

for the case $\mu = 0.235$ are shown in figure 1 with experimental data [11, 12] and with the curves given by other theories, i.e. K41 [7], log normal [13–15], β-model [16], p-model [5, 8] and log Poisson [17]. The asymptotic behaviour of ζ_m for $m \to \infty$ has a log term appearing in (21), i.e.

$$\zeta_m \to \alpha_{\min}\bar{q} - \delta\tau(m/3). \tag{30}$$

The curve (29) given by the present analysis successfully explains experimental data. Note that there is no fitting parameter.

The scaling exponent ζ_m for several values of μ listed in table 1 are shown in figure 2. Comparing the curves with experimental data, we conjecture that the value of the intermittency exponent μ for the fully developed turbulence can be narrowed down to $\mu = 0.235 \pm 0.015$.

The multifractal spectrum $f_{\mathrm{T}}(\alpha)$ of the present approach and $f_{\mathrm{B}}(\alpha)$ of the p-model are shown in figure 3 for $\mu = 0.235$. Note that $f_{\mathrm{T}}(\alpha) < 0$ for $\alpha > \alpha_{+}$, $\alpha < \alpha_{-}$.

The present analysis strongly indicates that the underlying statistics of fully developed turbulence is that of the Tsallis ensemble. The existence of a log term in the asymptotic expression of the scaling exponent (30) is one of the new features representing characteristics of Tsallis statistics. Experimental verification of this feature is highly desirable. With the proposed multifractal spectrum (16), we can investigate further the underlying dynamics supporting Tsallis statistics. This may provide us with a further understanding of turbulence in connection with the excess turbulent entropy which can be related to the pseudo-additivity property (11). Incorporation of skewness into the present approach may be one attractive future problem. We noticed that another [18] application to turbulence will be soon published using Tsallis statistics. The approach is, however, somewhat different and will be the subject of future comparison. These future problems will be reported elsewhere.

Let us close this letter by noting the case $q \to 1-$. We obtain $\alpha_0 = X = 2$, $f_{\mathrm{T}} = -\alpha(\alpha - 4)/4$ and $\tau(\bar{q}) = (1 - \bar{q})^2$, giving $\mu = 2$. Although the case $q = 1$ corresponds to a Gaussian distribution, it is not the same as the log-normal model [8]. From the value of μ, we can conclude that the case $q = 1$ for fully developed turbulence may not be realized in nature.

The authors would like to thank Professor C Tsallis for his appropriate comments and continuous encouragement, and Professor T Nakano, Professor T Goto and Professor S Kida for their fruitful comments.

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